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# The second approximation to mass transport in cnoidal waves

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A second approximation is developed for the mass-transport velocity within the bottom boundary layer of cnoidal waves progressing over a smooth horizontal bed. Mass-transport profiles through the boundary layer are obtained by considering terms of up to third order in the perturbation parameter. A comparison with results based on a first approximation indicates that the effect of the third-order terms is to predict a smaller mass-transport velocity and that this difference is generally significant, particularly for waves extending to the intermediate depth range. The predicted correction to the first approximation is qualitatively supported by experimental evidence.

## 1. Introduction

The mass transport, or particle drift velocity, in cnoidal waves propagating over a smooth horizontal bed was first calculated for an inviscid fluid by Le Méhauté (1968) on the basis of Laitone's (1960) cnoidal wave theory. More recently, Isaacson (1976) has derived an alternative expression for this case and has also investigated the effect of fluid viscosity on the mass-transport velocity within the laminar boundary layer at the bed.

The above studies were both carried out to a first approximation by considering terms of up to second order in the perturbation parameter. When higher-order terms are included, the net drift of any particle over a wave period depends not only on the initial location of the particle, but also on the initial instant within the wave cycle. However, by defining the average mass transport as the average of such drifts for all instants throughout the cycle, the mass-transport velocity may be obtained to a higher degree of approximation. In this manner, Sleath (1972) has derived a second approximation to the mass transport within the bottom boundary layer of Stokes waves.

In the present paper, the previous study (Isaacson 1976) is extended to the determination of a second approximation to the mass transport within the bottom boundary layer for the case of cnoidal waves. The results indicate a smaller mass-transport velocity than the prediction based on a first approximation and that this difference is generally appreciable, particularly for waves extending to the intermediate depth range. Available measurements of the mass-

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transport velocity near the bed correspond more closely to the predictions of the second approximation than to those of the first.

## 2. Theoretical development

It is assumed that the waves are steady and two-dimensional and that they propagate over a smooth horizontal bed. The water is assumed incompressible. Apart from the steady component of velocity, the motion outside the boundary layer is taken from Laitone's (1960) cnoidal wave theory applied at the bed. The notation previously used by the present author (1976) is adopted here and as before it is convenient to work with variables which are dimensionless in the extended set of dimensions $MXYT$ and defined as

$$\left.\begin{aligned} \xi = kx, \quad \eta = y/\delta, \quad u' = ku/\omega, \quad v' = v/\omega\delta, \\ U' = kU/\omega, \quad \tau = \omega t. \end{aligned}\right\} \tag{2.1}$$

Since the boundary-layer thickness is very much smaller than the typical lengths of the wave motion, the Prandtl boundary-layer equations are applicable to the motion within the boundary layer. These may be written in the non-dimensional form

$$\left.\begin{aligned} \frac{\partial u'}{\partial \tau} + u'\frac{\partial u'}{\partial \xi} + v'\frac{\partial u'}{\partial \eta} = \frac{\partial U'}{\partial \tau} + U'\frac{\partial U'}{\partial \xi} + \frac{1}{2}\frac{\partial^2 u'}{\partial \eta^2}, \\ \partial u'/\partial \xi + \partial v'/\partial \eta = 0, \end{aligned}\right\} \tag{2.2}$$

subject to the boundary conditions $u' = 0$ and $v' = 0$ at $\eta = 0$ and $u' = U'$ as $\eta \to \infty$. It was pointed out by Isaacson (1976) that the neglect of the term $V(\partial U/\partial y)$ in the boundary-layer equations is essentially a boundary-layer approximation and does not depend on the perturbation procedure to be introduced. We now assume that $u'$, $v'$ and $U'$ may all be expanded as asymptotic power series in a perturbation parameter $\epsilon$ in the form

$$f = \sum_{n=1}^{\infty} \epsilon^n f_n. \tag{2.3}$$

Substituting (2.3) into (2.2) and collecting successive powers of $\epsilon$, we obtain to first order

$$\left.\begin{aligned} \frac{\partial u_1'}{\partial \tau} - \frac{1}{2}\frac{\partial^2 u_1'}{\partial \eta^2} = \frac{\partial U_1'}{\partial \tau}, \\ \partial u_1'/\partial \xi + \partial v_1'/\partial \eta = 0, \end{aligned}\right\} \tag{2.4}$$

with

$$\left.\begin{aligned} u_1' = 0, \quad v_1' = 0 \quad \text{at} \quad \eta = 0, \\ u_1' = U_1' \quad \text{as} \quad \eta \to \infty. \end{aligned}\right\} \tag{2.5}$$

To second order

$$\left.\begin{aligned} \frac{\partial u_2'}{\partial \tau} - \frac{1}{2}\frac{\partial^2 u_2'}{\partial \eta^2} = -\left(u_1'\frac{\partial u_1'}{\partial \xi} + v_1'\frac{\partial u_1'}{\partial \eta}\right) + \frac{\partial U_2'}{\partial \tau} + U_1'\frac{\partial U_1'}{\partial \xi}, \\ \partial u_2'/\partial \xi + \partial v_2'/\partial \eta = 0, \end{aligned}\right\} \tag{2.6}$$

with

$$\left.\begin{aligned} u_2' = 0, \quad v_2' = 0 \quad \text{at} \quad \eta = 0, \\ u_2' = U_2', \quad \partial u_2'/\partial \eta = 0 \quad \text{as} \quad \eta \to \infty. \end{aligned}\right\} \tag{2.7}$$

At third order we shall require only the terms independent of time. Using an overbar to denote a temporal mean over the interval $t_0$ to $t_0+T$, where $T$ is the wave period, the time-averaged third-order terms give for $\overline{u_3'}$

$$-\frac{1}{2}\frac{\partial^2 \overline{u_3'}}{\partial\eta^2} = -\left(\overline{u_1'\frac{\partial u_2'}{\partial\xi}}+\overline{u_2'\frac{\partial u_1'}{\partial\xi}}+\overline{v_1'\frac{\partial u_2'}{\partial\eta}}+\overline{v_2'\frac{\partial u_1'}{\partial\eta}}\right)+\overline{U_1'\frac{\partial U_2'}{\partial\xi}}+\overline{U_2'\frac{\partial U_1'}{\partial\xi}}, \quad (2.8)$$

with

$$\left.\begin{aligned}\overline{u_3'} &= 0 \quad \text{at} \quad \eta = 0,\\ \partial\overline{u_3'}/\partial\eta &= 0 \quad \text{as} \quad \eta\to\infty.\end{aligned}\right\} \quad (2.9)$$

The terms $\overline{\partial u_3'/\partial\tau}$ and $\overline{\partial U_3'/\partial\tau}$ have been omitted from (2.8) since the motion is assumed periodic in time and therefore

$$\overline{\partial u_3'/\partial\tau} = [u_3']_{\tau_0}^{\tau_0+2\pi} = 0 \quad (2.10)$$

(where $\tau_0 = \omega t_0$). Similarly

$$\overline{\partial U_3'/\partial\tau} = 0. \quad (2.11)$$

Moreover, boundary conditions which specify that $\partial u_2'/\partial\eta$ and $\partial\overline{u_3'}/\partial\eta$ are zero as $\eta\to\infty$ have been introduced in (2.7) and (2.9) respectively. These additional conditions are necessary to obtain complete solutions for $u_2'$ and $\overline{u_3'}$ and correspond physically to there being no source of rapid variation of the steady velocity with $y$ outside the boundary layer (Batchelor 1967, p. 360). Also $\partial u_2'/\partial\eta$ itself tends to zero as $\eta\to\infty$ since the fluctuating component of $\partial u_2'/\partial\eta$ ($\eta\to\infty$) is given by the value of $\partial U_2'/\partial y$ at the bed in a wholly irrotational motion and is zero.

In order to solve (2.4), (2.6) and (2.8), we shall require the description of $U_1$ and $U_2$ given by cnoidal theory. It is convenient to choose the perturbation parameter $\epsilon$ to be equal to the ratio $H/h$, where $H$ is the wave height and $h$ the trough depth, since the results of cnoidal theory may then be applied directly. Thus, with sign changes accounting for the reversal in the direction defining $x$, we have from Le Méhauté (1968)

$$U_1 = (gh)^{\frac{1}{2}}\left\{\operatorname{cn}^2(q)-\left(\frac{\gamma-\kappa'^2}{\kappa^2}\right)\right\}, \quad (2.12)$$

$$\begin{aligned}U_2 = -(gh)^{\frac{1}{2}}\Bigg\{&\frac{\kappa'^2}{6\kappa^4}(3\kappa^2+2)+\left(\frac{7\kappa^2-2}{4\kappa^2}\right)\operatorname{cn}^2(q)\\ &-\tfrac{5}{4}\operatorname{cn}^4(q)-\frac{\gamma}{12\kappa^4}(\kappa^2+4)+\frac{3}{4\kappa^2}\left[\left(\frac{y}{h}\right)^2-1\right]\\ &\times[\kappa'^2+2(2\kappa^2-1)\operatorname{cn}^2(q)-3\kappa^2\operatorname{cn}^4(q)]\Bigg\}. \quad (2.13)\end{aligned}$$

Here cn is the Jacobian elliptic function, with argument $q = K(\kappa)(kx-\omega t)/\pi$ and modulus $\kappa$, $\gamma$ is the ratio of the complete elliptic integrals $E(\kappa)/K(\kappa)$ and $\kappa'^2 = 1-\kappa^2$. In fact (2.13) derives from the assumption that the mean Eulerian velocity is zero for the irrotational motion, and putting $y = 0$ we obtain the fluctuating component of $U_2$ at the boundary-layer edge.

It is convenient to represent $U_1'$ and $U_2'$ at the boundary-layer edge as complex Fourier series:

$$U_1' = \sum_{n=-\infty}^{\infty} A_n e^{in\theta}, \quad A_{-n} = A_n^*, \quad A_0 = 0, \tag{2.14}$$

$$U_2' = \sum_{n=-\infty}^{\infty} B_n e^{in\theta}, \quad B_{-n} = B_n^*, \tag{2.15}$$

where an asterisk denotes the complex conjugate and

$$\theta = \xi - \tau = kx - \omega t. \tag{2.16}$$

We now define

$$A_n' = (c/(gh)^{\frac{1}{2}}) A_n, \quad B_n' = (c/(gh)^{\frac{1}{2}}) B_n. \tag{2.17}$$

where $c$ is the wave speed ($=\omega/k$). Then $A_n'$ and $B_n'$ depend only on the modulus $\kappa$ and may be determined numerically for all $n$ by expressing the Jacobian elliptic functions occurring in (2.12) and (2.13) as Fourier series. This was carried out by Isaacson (1976) for $A_n'$ and we shall here derive a corresponding expression for $B_n'$. In order to do so the following identities are introduced. If $F$ and $G$ are any two periodic functions

$$F = \sum_{n=-\infty}^{\infty} f_n e^{in\theta}, \quad G = \sum_{n=-\infty}^{\infty} g_n e^{in\theta} \tag{2.18}$$

with no restriction on $f_n$ and $g_n$, then

$$FG = \sum_{n=-\infty}^{\infty} \left( \sum_{m=-\infty}^{\infty} f_m g_{n-m} \right) e^{in\theta}, \quad \overline{FG} = \sum_{n=-\infty}^{\infty} f_n g_{-n}. \tag{2.19}$$

We shall also require a further identity eventually:

$$\overline{F\frac{\partial G}{\partial \xi}} + \overline{G\frac{\partial F}{\partial \xi}} = \sum_{n=-\infty}^{\infty} -inf_n g_{-n} + \sum_{n=-\infty}^{\infty} -inf_{-n} g_n = 0. \tag{2.20}$$

Returning now to the evaluation of $B_n'$, we put $y = 0$ in (2.13) as mentioned previously and obtain

$$\sum_{n=-\infty}^{\infty} B_n' e^{in\theta} = \left\{ \left( \frac{5\kappa^2 - 4}{4\kappa^2} \right) \mathrm{cn}^2(q) - \mathrm{cn}^4(q) \right\} - \left\{ \left( \frac{5\kappa^2 - 4}{4\kappa^2} \right) \overline{\mathrm{cn}^2(q)} - \overline{\mathrm{cn}^4(q)} \right\}. \tag{2.21}$$

Now from (2.12), (2.14) and (2.17), $\mathrm{cn}^2(q)$ is given in terms of $A_n'$ as

$$\mathrm{cn}^2(q) = \left( \frac{\gamma - \kappa'^2}{\kappa^2} \right) + \sum_{n=-\infty}^{\infty} A_n' e^{in\theta}. \tag{2.22}$$

Squaring (2.22) and using the result (2.19) gives

$$\mathrm{cn}^4(q) = \left( \frac{\gamma - \kappa'^2}{\kappa^2} \right)^2 + 2\left( \frac{\gamma - \kappa'^2}{\kappa^2} \right) \sum_{n=-\infty}^{\infty} A_n' e^{in\theta} + \sum_{n=-\infty}^{\infty} \left( \sum_{m=-\infty}^{\infty} A_m' A_{n-m}' \right) e^{in\theta}. \tag{2.23}$$

Substituting (2.22) and (2.23) into (2.21) and since $A_{-n}' = A_n'$, we obtain after some rearrangement

$$\left.\begin{aligned} B_n' &= -\left\{ \frac{1}{4\kappa^2}(8\gamma + 3\kappa^2 - 4) A_n' + \sum_{m=1}^{n-1} A_m' A_{n-m}' + 2\sum_{m=1}^{\infty} A_m' A_{n+m}' \right\}, \\ B_{-n}' &= B_n', \quad B_0' = 0. \end{aligned}\right\} \tag{2.24}$$

Hence both $A'_n$ and $B'_n$ may be determined numerically for any given value of the modulus $\kappa$.

We have yet to consider the definition of the mass-transport velocity $U_M$. It is assumed that $U_M$ may be expressed in the form (2.3) and that there is no first-order steady motion, i.e. $U_{M1} = 0$. The first approximation to the mass-transport velocity, $U_{M2}$, was obtained for the present case by Isaacson (1976):

$$\frac{U_{M2}}{gh/c} = \sum_{n=1}^{\infty} A_n'^2\{5+3\exp(-2n^{\frac{1}{2}}\eta) - 8\exp(-n^{\frac{1}{2}}\eta)\cos(n^{\frac{1}{2}}\eta)\}. \qquad (2.25)$$

To third order, the net drift of any fluid particle over a wave period depends on the initial instant $t_0$ considered. A general expression for this third-order drift velocity $U_{m3}(t_0)$ was given by Sleath (1972) and essentially derives from a Taylor-series expansion of the velocity of a particle, together with the power-series assumption (2.3). It applies to the present case also and, in terms of the dimensionless variables used here, reduces to

$$\begin{aligned}\frac{U_{m3}(t_0)}{c} = {} & \overline{u_3'} + \overline{\frac{\partial u_2'}{\partial \xi}\int u_1'\,d\tau} + \overline{\frac{\partial u_1'}{\partial \xi}\int u_2'\,d\tau} \\ & + \overline{\frac{\partial u_1'}{\partial \xi}\int\left(\frac{\partial u_1'}{\partial \xi}\int u_1'\,d\tau\right)d\tau} + \overline{\frac{\partial u_1'}{\partial \xi}\int\left(\frac{\partial u_1'}{\partial \eta}\int v_1'\,d\tau\right)d\tau} \\ & + \overline{\frac{1}{2}\frac{\partial^2 u_1'}{\partial \xi^2}\left(\int u_1'\,d\tau\right)^2} + \overline{\frac{\partial u_2'}{\partial \eta}\int v_1'\,d\tau} \\ & + \overline{\frac{\partial u_1'}{\partial \eta}\int v_2'\,d\tau} + \overline{\frac{\partial u_1'}{\partial \eta}\int\left(\frac{\partial v_1'}{\partial \xi}\int u_1'\,d\tau\right)d\tau} \\ & + \overline{\frac{\partial u_1'}{\partial \eta}\int\left(\frac{\partial v_1'}{\partial \eta}\int v_1'\,d\tau\right)d\tau} + \overline{\frac{1}{2}\frac{\partial^2 u_1'}{\partial \eta^2}\left(\int v_1'\,d\tau\right)^2} \\ & + \overline{\frac{\partial^2 u_1'}{\partial \xi\,\partial \eta}\left(\int u_1'\,d\tau\right)\left(\int v_1'\,d\tau\right)}. \end{aligned} \qquad (2.26)$$

All the above integrals are taken between the limits $\tau_0$ and $\tau$. We shall in fact require the average third-order mass-transport velocity, which is defined as

$$U_{M3} = \frac{1}{T}\int_0^T U_{m3}(t_0)\,dt_0. \qquad (2.27)$$

For cnoidal wave motion it is convenient to non-dimensionalize velocities with respect to $(gh)^{\frac{1}{2}}$ and to express the mass transport in terms of $\epsilon$ and $\kappa$ only. To the present order of approximation, we then write

$$\frac{U_M}{(gh)^{\frac{1}{2}}} = \epsilon^2\frac{U_{M2}}{(gh)^{\frac{1}{2}}} + \epsilon^3\frac{U_{M3}}{(gh)^{\frac{1}{2}}}. \qquad (2.28)$$

Now cnoidal theory gives the wave speed $c$ as $(gh)^{\frac{1}{2}}(1+O[\epsilon])$ and consequently the factor $c/(gh)^{\frac{1}{2}}$ in the first approximation (2.25) will introduce terms of order $\epsilon^3$ and higher. Thus we omit the factor $c/(gh)^{\frac{1}{2}}$ from (2.25) and write for the first approximation

$$\frac{U_{M2}}{(gh)^{\frac{1}{2}}} = \sum_{n=1}^{\infty} A_n'^2\{5+3\exp(-2n^{\frac{1}{2}}\eta) - 8\exp(-n^{\frac{1}{2}}\eta)\cos(n^{\frac{1}{2}}\eta)\}; \tag{2.29}$$

the resulting term in $\epsilon^3$ will be considered later.

We now proceed to derive solutions to (2.4), (2.6) and (2.8), intending eventually to substitute these into (2.26) and (2.27) and obtain an expression for $U_{M3}$. The solution to (2.4) subject to the boundary conditions (2.5) was given by Isaacson (1976):

$$u_1' = \sum_{n=-\infty}^{\infty} A_n\{1-\exp(-\alpha_n\eta)\}e^{in\theta}, \tag{2.30}$$

$$v_1' = \sum_{n=-\infty}^{\infty} -inA_n\left(\eta + \frac{\exp(-\alpha_n\eta)}{\alpha_n} - \frac{1}{\alpha_n}\right)e^{in\theta}, \tag{2.31}$$

where

$$\alpha_n = (1-i)n^{\frac{1}{2}}. \tag{2.32}$$

We now consider the solution to (2.6) and substitute the expressions for $U_1'$, $U_2'$, $u_1'$ and $v_1'$, given by (2.14), (2.15), (2.30) and (2.31) respectively, into the right-hand side of (2.6) and use the identity (2.19) to obtain

$$\begin{aligned}\frac{\partial u_2'}{\partial \tau} - \frac{1}{2}\frac{\partial^2 u_2'}{\partial \eta^2} = \sum_{n=-\infty}^{\infty}\Bigg\{-inB_n + \sum_{m=-\infty}^{\infty} A_m A_{n-m}\Bigg[& i(n-m)\exp(-\alpha_m\eta) \\ &+ \left(i(n-m) - im\frac{\alpha_{n-m}}{\alpha_m}\right)\exp(-\alpha_{n-m}\eta) \\ &- \left(i(n-m) - im\frac{\alpha_{n-m}}{\alpha_m}\right)\exp(-(\alpha_m+\alpha_{n-m})\eta) \\ &+ im\alpha_{n-m}\eta\exp(-\alpha_{n-m}\eta)\Bigg]\Bigg\}e^{in\theta}. \end{aligned} \tag{2.33}$$

The solutions for $u_2'$ and, consequently, for $v_2'$ which satisfy the boundary conditions (2.7) are then found to be

$$\begin{aligned}u_2' = \sum_{n=-\infty}^{\infty}\Bigg\{& B_n[1-\exp(-\alpha_n\eta)] + \sum_{m=-\infty}^{\infty} A_m A_{n-m}[\sigma_1\exp(-\alpha_m\eta) \\ &+ \sigma_2\exp(-\alpha_{n-m}\eta) + \sigma_3\exp\{-(\alpha_m+\alpha_{n-m})\eta\} + \sigma_4\eta\exp(-\alpha_{n-m}\eta) \\ &+ \sigma_5\exp(-\alpha_n\eta)]\Bigg\}e^{in\theta}, \end{aligned} \tag{2.34}$$

$$\begin{aligned}v_2' = \sum_{n=-\infty}^{\infty} -in\Bigg\{& B_n\left(\eta + \frac{\exp(-\alpha_n\eta)}{\alpha_n} - \frac{1}{\alpha_n}\right) - \sum_{m=-\infty}^{\infty} A_m A_{n-m}\Bigg[\frac{\sigma_1}{\alpha_m}\exp(-\alpha_m\eta) \\ &+ \frac{\sigma_2}{\alpha_{n-m}}\exp(-\alpha_{n-m}\eta) + \frac{\sigma_3}{(\alpha_m+\alpha_{n-m})}\exp\{-(\alpha_m+\alpha_{n-m})\eta\} \\ &+ \frac{\sigma_4}{\alpha_{n-m}}\eta\exp(-\alpha_{n-m}\eta) + \frac{\sigma_4}{\alpha_{n-m}^2}\exp(-\alpha_{n-m}\eta) \\ &+ \frac{\sigma_5}{\alpha_n}\exp(-\alpha_n\eta) - \sigma_6\Bigg]\Bigg\}e^{in\theta}, \end{aligned} \tag{2.35}$$

where $$\sigma_1 = -1, \quad \sigma_2 = \left(\frac{n-m}{m}\right) + \frac{\alpha_{n-m}}{\alpha_m}, \quad \sigma_3 = \frac{1}{2}\left(1 - \frac{\alpha_{n-m}}{\alpha_m}\right), \tag{2.36)–(2.38}$$

$$\sigma_4 = -\alpha_{n-m}, \quad \sigma_5 = \sigma_2 - \left(\frac{n-m}{m}\right), \tag{2.39), (2.40}$$

$$\sigma_6 = \frac{\sigma_1}{\alpha_m} + \frac{\sigma_2}{\alpha_{n-m}} + \frac{\sigma_3}{(\alpha_m + \alpha_{n-m})} + \frac{\sigma_4}{\alpha_{n-m}^2} + \frac{\sigma_5}{\alpha_n}. \tag{2.41}$$

It may be noted that, as expected, those terms in (2.34) corresponding to $n = 0$ reduce to the solution for $\overline{u_2'}$ given by Isaacson (1976).

We are now in a position to derive the solution to (2.8) for $\overline{u_3'}$. Initially the identity (2.20) is applied to $u_1'$, $u_2'$ and $U_1'$, $U_2'$ in turn:

$$\overline{u_1' \frac{\partial u_2'}{\partial \xi}} + \overline{u_2' \frac{\partial u_1'}{\partial \xi}} = 0 \tag{2.42}$$

and

$$\overline{U_1' \frac{\partial U_2'}{\partial \xi}} + \overline{U_2' \frac{\partial U_1'}{\partial \xi}} = 0. \tag{2.43}$$

Thus the equation describing $\overline{u_3'}$ reduces to

$$\frac{\partial^2 \overline{(u_3')}}{\partial \eta^2} = 2\left(\overline{v_1' \frac{\partial u_2'}{\partial \eta}} + \overline{v_2' \frac{\partial u_1'}{\partial \eta}}\right). \tag{2.44}$$

Substituting the expressions obtained for $u_1'$, $v_1'$, $u_2'$ and $v_2'$, (2.44) is transformed to a lengthy equation of the form

$$\begin{aligned}\frac{\partial^2 \overline{(u_3')}}{\partial \eta^2} = 2 \sum_{n=-\infty}^{\infty} inA_n \Bigg\{ B_n \left[ \sum_{l=1}^{4} \Lambda_l \exp(-\lambda_l \eta) + \sum_{l=5}^{6} \Lambda_l \eta \exp(-\lambda_l \eta) \right] \\ + \sum_{m=-\infty}^{\infty} A_m A_{n-m} \left[ \sum_{l=7}^{22} \Lambda_l \exp(-\lambda_l \eta) \right. \\ \left. + \sum_{l=23}^{30} \Lambda_l \eta \exp(-\lambda_l \eta) + \Lambda_{31} \eta^2 \exp(-\lambda_{31} \eta) \right] \Bigg\},\end{aligned} \tag{2.45}$$

where $\Lambda_l$ and $\lambda_l$ are known and generally complex functions of $m$ and $n$ or of $n$ alone. Integrating twice with respect to $\eta$ and applying the boundary conditions (2.9) gives

$$\begin{aligned}\overline{u_3'} = 2 \sum_{n=-\infty}^{\infty} inA_n \Bigg\{ B_n \left[ \sum_{l=1}^{4} \frac{\Lambda}{\lambda^2} (e^{-\lambda\eta} - 1) + \sum_{l=5}^{6} \left( \frac{\Lambda}{\lambda^2} \eta e^{-\lambda\eta} + \frac{2\Lambda}{\lambda^3} (e^{-\lambda\eta} - 1) \right) \right] \\ + \sum_{m=-\infty}^{\infty} A_m A_{n-m} \left[ \sum_{l=7}^{22} \frac{\Lambda}{\lambda^2} (e^{-\lambda\eta} - 1) + \sum_{l=23}^{30} \left( \frac{\Lambda}{\lambda^2} \eta e^{-\lambda\eta} + \frac{2\Lambda}{\lambda^3} (e^{-\lambda\eta} - 1) \right) \right. \\ \left. + \left( \frac{\Lambda}{\lambda^2} \eta^2 e^{-\lambda\eta} + \frac{4\Lambda}{\lambda^3} \eta e^{-\lambda\eta} + \frac{6\Lambda}{\lambda^4} (e^{-\lambda\eta} - 1) \right)_{l=31} \right] \Bigg\},\end{aligned} \tag{2.46}$$

in which the subscript $l$ has been omitted for clarity.

By similar reasoning to that given previously, since $c$ is $(gh)^{\frac{1}{2}}(1 + O[\epsilon])$ the factor $c/(gh)^{\frac{1}{2}}$ multiplying various terms in the above equation will introduce terms of order $\epsilon^4$ and higher. Thus the replacement of $A_n$, $B_n$ and $u_3'$ by $A_n'$, $B_n'$

and $u_3/(gh)^{\frac{1}{2}}$ respectively only gives rise to additional terms of higher order in $\epsilon$ and therefore does not affect the present approximation. The equation will then describe the profile $\overline{u}_3/(gh)^{\frac{1}{2}}(\eta)$, which depends on the coefficients $A'_n$ and $B'_n$ and thus only on the modulus $\kappa$. Now although (2.46) may be simplified further to reduce the number of terms, it is convenient to compute the profile $\overline{u}_3/(gh)^{\frac{1}{2}}(\eta)$ for any given value of $\kappa$ using the equation in its present form, but taking twice the real part of the overall summation for positive values of $n$ only.

To determine the third-order mass-transport velocity $U_{M3}$, we also need to calculate the remaining terms in (2.26), which involve $u'_1$, $v'_1$, $u'_2$ and $v'_2$, and then use (2.27). This should be straightforward since expressions for all these velocities have been obtained and after some calculation the average of these terms with respect to $t_0$ eventually gives

$$\begin{aligned}\frac{1}{T}\int_0^T\left(\frac{U_{m3}}{c}-\overline{u'_3}\right)dt_0 = \sum_{n=-\infty}^{\infty} A_n\Bigg\{B_n\,[2f_{-n}(\eta)f_n(\eta)+h_{-n}(\eta)\,g_n(\eta)+h_n(\eta)\,g_{-n}(\eta)]\\ +\sum_{m=-\infty}^{\infty} A_m A_{n-m}\Bigg\{\left(\frac{n^2-2m^2}{2m(n-m)}\right)f_{-n}(\eta)f_m(\eta)f_{n-m}(\eta)\\ -\tfrac{1}{2}\alpha_{-n}h_{-n}(\eta)\,g_m(\eta)\,g_{n-m}(\eta)+f_{-n}(\eta)\,h_m(\eta)\,g_{n-m}(\eta)+2h_{-n}(\eta)f_m(\eta)\,g_{n-m}(\eta)\\ +\sigma_1\exp(-\alpha_m\eta)\,[2f_{-n}(\eta)-\alpha_m^{-1}h_{-n}(\eta)-\alpha_m g_{-n}(\eta)]\\ +\sigma_2\exp(-\alpha_{n-m}\eta)\,[2f_{-n}(\eta)-\alpha_{n-m}^{-1}h_{-n}(\eta)-\alpha_{n-m}g_{-n}(\eta)]\\ +\sigma_3\exp\{-(\alpha_m+\alpha_{n-m})\,\eta\}\,[2f_{-n}(\eta)-(\alpha_m+\alpha_{n-m})^{-1}h_{-n}(\eta)\\ -(\alpha_m+\alpha_{n-m})\,g_{-n}(\eta)]\\ +\sigma_4\exp(-\alpha_{n-m}\eta)\,[\eta\,(2f_{-n}(\eta)-\alpha_{n-m}^{-1}h_{-n}(\eta)-\alpha_{n-m}g_{-n}(\eta))\\ -\alpha_{n-m}^{-2}h_{-n}(\eta)+g_{-n}(\eta)]\\ +\sigma_5\exp(-\alpha_n\eta)\,[2f_{-n}(\eta)-\alpha_n^{-1}h_{-n}(\eta)-\alpha_n g_{-n}(\eta)]+\sigma_6 h_{-n}(\eta)\Bigg\}\Bigg\},\end{aligned} \tag{2.47}$$

where

$$\left.\begin{aligned} f_n(\eta) &= 1-\exp(-\alpha_n\eta),\\ g_n(\eta) &= \eta+\alpha_n^{-1}\exp(-\alpha_n\eta)-\alpha_n^{-1},\\ h_n(\eta) &= \alpha_n\exp(-\alpha_n\eta).\end{aligned}\right\} \tag{2.48}$$

Again, if $A_n$ and $B_n$ are replaced by $A'_n$ and $B'_n$, the right side of (2.47) may be evaluated for any given $\kappa$. The sum of (2.46) and (2.47) then gives $U_{M3}/(gh)^{\frac{1}{2}}(\eta)$ as a function of $\kappa$ only. However, when the factor $c/(gh)^{\frac{1}{2}}$ was applied to the first approximation (2.25) to obtain the form (2.29), it was mentioned that an additional term in $\epsilon^3$ arises. Now from Laitone (1960), the wave speed is given by

$$c/(gh)^{\frac{1}{2}} = 1+\epsilon(1-2\gamma)/2\kappa^2+O[\epsilon^2]. \tag{2.49}$$

The additional term in $\epsilon^3$ resulting from the modification to the form of the first approximation is found by substituting (2.49) into (2.25):

$$\frac{U_c}{(gh)^{\frac{1}{2}}} = \left(\frac{2\gamma-1}{2\kappa^2}\right)\sum_{n=1}^{\infty} A_n'^2\{5+3\exp(-2n^{\frac{1}{2}}\eta)-8\exp(-n^{\frac{1}{2}}\eta)\cos(n^{\frac{1}{2}}\eta)\}. \tag{2.50}$$

When this is added to the other components of the third-order mass transport already described, we may obtain $U_{M3}/(gh)^{\frac{1}{2}}$. The calculations have been carried

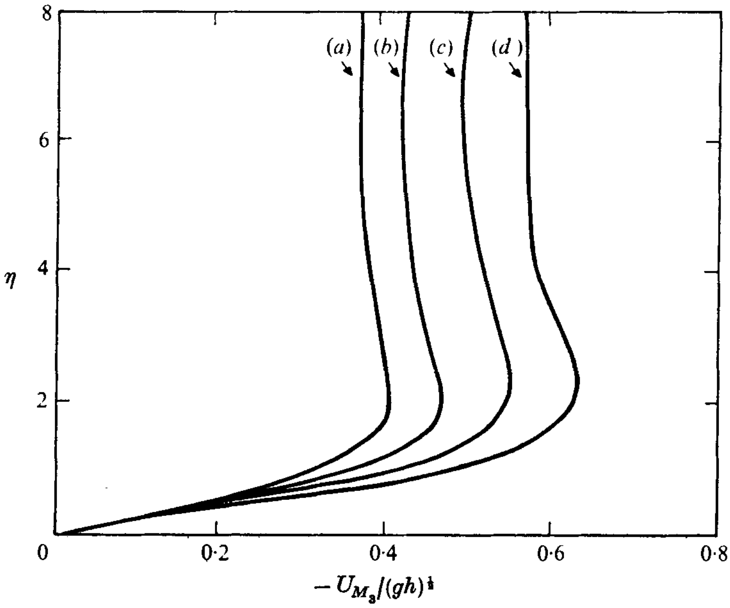


FIGURE 1. Third-order mass-transport velocity distributions through the bottom boundary layer for various values of the modulus $\kappa$: (*a*) $\kappa = 0{\cdot}999$, (*b*) $\kappa = 0{\cdot}99$, (*c*) $\kappa = 0{\cdot}95$, (*d*) $\kappa = 0{\cdot}90$.

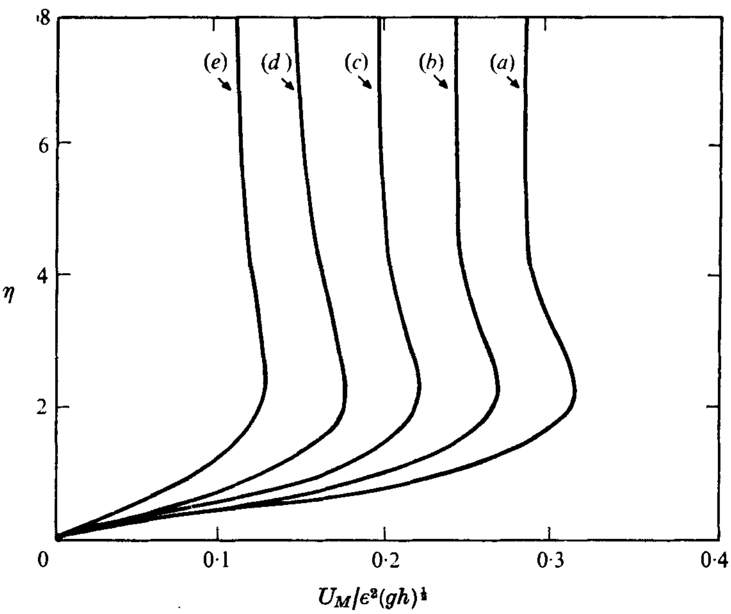


FIGURE 2. Mass-transport velocity distributions through the bottom boundary layer for $\kappa = 0{\cdot}99$ and various values of $\epsilon$: (*a*) first approximation, $U_M = \epsilon^2 U_{M_2}$, (*b*) $\epsilon = 0{\cdot}1$, (*c*) $\epsilon = 0{\cdot}2$, (*d*) $\epsilon = 0{\cdot}3$, (*e*) $\epsilon = 0{\cdot}4$.

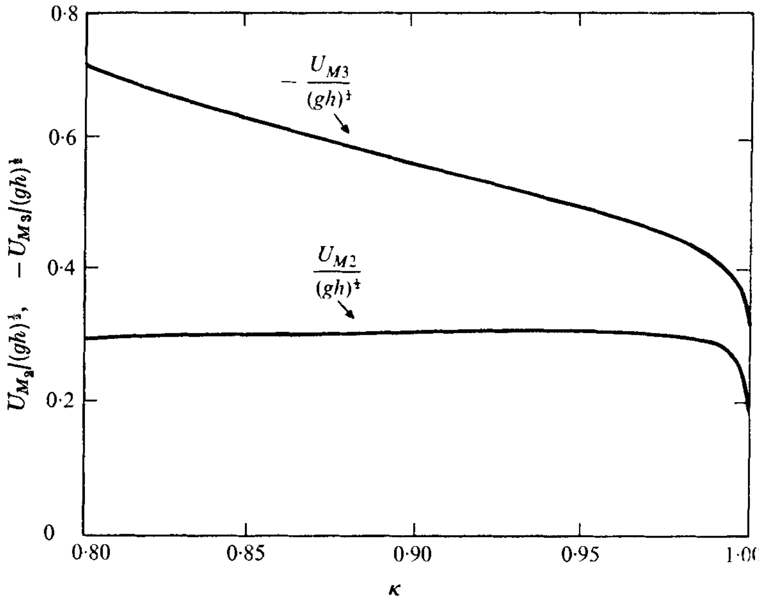


FIGURE 3. Second- and third-order components of the mass-transport velocity at the outer edge of the bottom boundary layer as functions of the modulus $\kappa$.

out on the University of Hawaii computer and some profiles of $U_{M3}/(gh)^{\frac{1}{2}}(\eta)$ for various values of $\kappa$ approaching unity are presented in figure 1.

## 3. Results and discussion

The effect of taking the mass-transport calculation to a second approximation is considered here with particular reference to (2.28). The distributions of $U_{M2}/(gh)^{\frac{1}{2}}$ and $U_M/\epsilon^2(gh)^{\frac{1}{2}}$ through the boundary layer for various values of $\epsilon$ are shown in figure 2 for the particular case $\kappa = 0{\cdot}99$. It may be noted that the smaller mass-transport velocity given by the second approximation persists throughout the boundary layer and thus any difference in the predicted mass-transport velocity at the boundary-layer edge is generally indicative of conditions within the boundary layer.

The mass-transport velocity at the outer edge of the boundary layer is in any case of particular interest and of some practical importance. The first approximation to this was derived by Isaacson (1976):

$$U_{M2}/(gh)^{\frac{1}{2}} = 5\{2\gamma(2-\kappa^2) - 3\gamma^2 - \kappa'^2\}/6\kappa^4. \tag{3.1}$$

The third-order component of the mass transport at the boundary-layer edge is found by letting $\eta \rightarrow \infty$ in all those terms comprising $U_{M3}/(gh)^{\frac{1}{2}}$. These include $\overline{U_3}/(gh)^{\frac{1}{2}}$, which is not zero for a viscous fluid, the Lagrangian terms given by (2.47) and finally the correction term arising from $U_{M2}/(gh)^{\frac{1}{2}}$. From (2.46), (2.47) and (2.50), $U_{M3}/(gh)^{\frac{1}{2}}$ may be expressed in a slightly simpler form for $\eta \rightarrow \infty$ and may be evaluated numerically as a function of $\kappa$. The variation of $U_{M2}/(gh)^{\frac{1}{2}}$ and $U_{M3}/(gh)^{\frac{1}{2}}$ at the outer edge of the boundary layer with $\kappa$ is presented in figure 3,

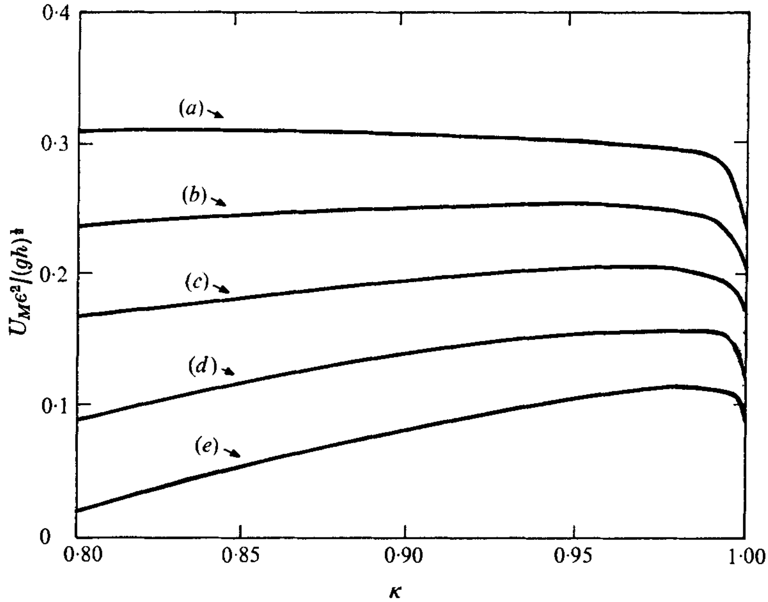


FIGURE 4. The mass-transport velocity at the outer edge of the bottom boundary layer as a function of the modulus $\kappa$ for various values of $\epsilon$: (a) first approximation, $U_M = \epsilon^2 U_{M_2}$, (b) $\epsilon = 0{\cdot}1$, (c) $\epsilon = 0{\cdot}2$, (d) $\epsilon = 0{\cdot}3$, (e) $\epsilon = 0{\cdot}4$.

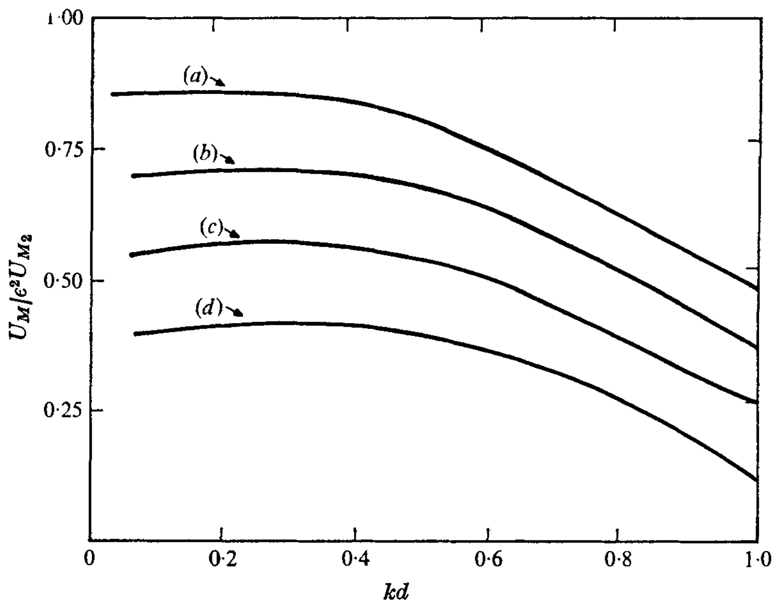


FIGURE 5. Ratio of the second to the first approximation of the mass-transport velocity at the outer edge of the bottom boundary layer as a function of the wave depth parameter $kd$ for various values of $\epsilon$: (a) $\epsilon = 0{\cdot}1$, (b) $\epsilon = 0{\cdot}2$, (c) $\epsilon = 0{\cdot}3$, (d) $\epsilon = 0{\cdot}4$.

which may conveniently be used to evaluate the mass transport for any particular case. Thus, for given wave parameters, values of $\epsilon$ and $\kappa$ may be obtained and then figure 3 and (2.28) readily permit an estimate of $U_M/(gh)^{\frac{1}{2}}$.

The significance of the second approximation in determining the mass transport at the boundary-layer edge may be assessed from figure 4, which shows the dependence on the modulus $\kappa$ of $U_{M2}/(gh)^{\frac{1}{2}}$ together with that of $U_M/\epsilon^2(gh)^{\frac{1}{2}}$ for various values of $\epsilon$. It is seen that the correction to the first approximation becomes appreciable as $\kappa$ decreases from unity, this corresponding to waves extending to the intermediate depth range.

More precisely, cnoidal theory gives the wave depth parameter $kd$, where $d$ is the mean depth, as

$$kd = \frac{\pi(3\epsilon)^{\frac{1}{2}}}{2\kappa K(\kappa)}\left\{1+\frac{\epsilon}{8\kappa^2}(8\gamma+\kappa^2-6)+O[\epsilon^2]\right\}. \tag{3.2}$$

Thus, from figure 4, we may obtain the fractional decrease from the first approximation resulting from the third-order terms as a function of the wave depth parameter $kd$. Figure 5 thus presents $U_M/\epsilon^2 U_{M2}$ as a function of $kd$ for various values of $\epsilon$ and shows clearly that the second approximation should generally be taken into account. Even for waves with $\epsilon = 0{\cdot}1$, the third-order terms cause a reduction of more than 10 % for waves of any depth, while for higher waves, say $\epsilon > 0{\cdot}2$, the reduction is always more than 20 %. For waves of intermediate depth ($kd > 0{\cdot}3$) the reduction from the first approximation becomes more severe as $kd$ increases. This is expected since it was mentioned by Isaacson (1976) that the first approximation tends to be an overestimate beyond the shallow-water range.

On the other hand, for very small values of $kd$, the mass-transport velocity also becomes small. In the limit $\kappa \to 1$ ($kd \to 0$), corresponding to the solitary-wave case, the result of zero mass-transport velocity is inappropriate and it is then the total displacement of fluid, rather than the displacement averaged over the wave period, that is of interest.

Brebner & Collins (1961) have measured the bottom mass-transport velocity over a range of conditions and the original data have been supplied to the author. It is appropriate to contrast comparisons of these data with the predictions of the first and second approximations respectively, and so to consider only conditions reasonably near the shallow-water range. Those data corresponding to $kd < 0{\cdot}7$ are analysed here.

For each set of values of $H$, $T$ and $d$, the corresponding values of $\epsilon$, $\kappa$ and $(gh)^{\frac{1}{2}}$ may be calculated numerically and thus the predicted values of the mass-transport velocity based on the first and second approximations, $\epsilon^2 U_{M2}$ and $U_M$ respectively, may be calculated. These are compared with the measured bottom velocity $U_b$ in figure 6, in which the ratios $U_b/\epsilon^2 U_{M2}$ and $U_b/U_M$ are plotted against the perturbation parameter $\epsilon$. Despite the extent of the scatter, which must be expected in this kind of experiment, trends showing $U_b/\epsilon^2 U_{M2}$ to decrease with $\epsilon$ to values below unity and $U_b/U_M$ to remain near unity and independent of $\epsilon$ are clearly evident. This observation is expected from the theory [see (2.28)], which indicates that the deviation from the first approximation increases with $\epsilon$, and that the correction introduced by taking the calculation to a second approxima-

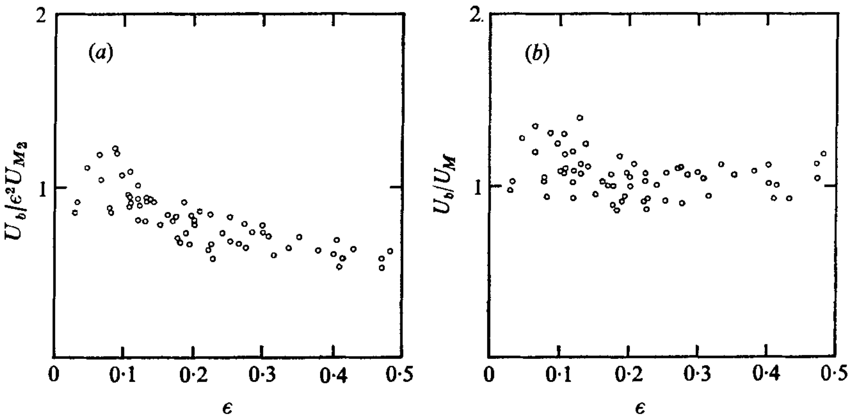


FIGURE 6. A comparison of the measured bottom velocity $U_b$ with the predictions of (*a*) the first approximation $\epsilon^2 U_{M_2}$ and (*b*) the second approximation $U_M$. The ratios $U_b/\epsilon^2 U_{M_2}$ and $U_b/U_M$ are plotted against $\epsilon$ and the figure is based on the data of Brebner & Collins (1961).

tion serves to restore the ratio of measured to predicted velocity closer to unity at greater values of $\epsilon$.

Over the range of depths considered, then, it appears from the above comparison that the second approximation may reasonably be used for waves of general height, whereas the first approximation is realistic only for waves with, say, $\epsilon \leqslant 0{\cdot}1$.

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